**Partial Derivatives And Applications**

We have learnt the rules, formulae and application of differentiation. Here we shall understand partial derivative and its application.

Let Z or f(x,y) be a function of two variables x and y. . If we keep y constant and differentiate f (assuming f is differentiable) with respect to the variable x, then we obtain the **partial derivative** of f with respect to x. It is denoted by



Similarly , if we keep x constant and differentiate f (assuming f is differentiable) with respect to the variable y, then we get the **partial derivative** of f with respect to y. It is denoted by



All rules and formulae of differentiation apply for finding partial derivatives.

### Example :

Find the partial derivatives fx and fy , if



Assume y is constant and differentiate with respect to x to obtain
 **=** 2xy + 2

Now assume x is constant and differentiate with respect to y to obtain

** =** x2 + 1 **( w**e treated x as a constant and differentiated w.r.t y**)**

We can find out second order partial derivatives by differentiating the above w.r.t x or y.

Consider another example :

Β = 4 logy + 8 log x

 = 8 / x

 = 4 / y

Now, let us understand what are homogeneous functions :

**Homogeneous Functions**:

Consider Given Z is a function of x and y. Z = f(x,y).

Multiply each variable with a constant β to get Z1 = f(βx, βy)

If we are able to get the result

**f(βx, βy) =** Z1 **= βn f(x,y)**

then we say that this function is homogeneous of degree n.

If n = 1, it is called a linear homogeneous function i.e. degree of homogeneity is 1.

Consider 4 different functions as below:

1. f(x,y) = Z = f(x,y) = 4x2 + y2

 2. f(x,y) = Z = 20 x2 + 23 y2

 3. f(x,y) = Z = 2xy3 + 3y4

4. Z= 4x + 6y

Consider the first function: f(x,y) = 4x2 + y2 . It is a homogeneous function as demonstrated below:

Multiply each variable by β:

f(βx,βy) = 4(βx)2 + (βy)2

f(βx,βy) = 4β2x2 + β2y2

Factoring out β2 we getf(βx,βy) = β2(4x2 + y2)

We get **f(βx,βy) = β2f(x,y)**

Thus we say that it is a homogeneous function with Degree of homogeneity =2

By same logic, we can test homogeneity of all functions listed above. 2nd function has degree of homogeneity of 2 and 3rd function is homogeneous with degree 4. Similarly, fourth function 4x + 6y is a homogeneous function with degree 1.

However ,a function Z = 2x+9 is not a homogeneous function**.**

**All first order partial derivatives of a homogeneous function will also be homogeneous with degree 1 less than that of the original function**.

For example: the first order partial derivatives for the function f(x, y) = 4x2 + y2  are both homogeneous with degree 1

fx = 8x

fy = 2y

**Euler’s Theorem**:

If a function Z is homogeneous of degree **n** and has continuous first and 2nd order derivatives (i.e if the function is differentiable) then according to Euler’s theorem ,

x. fx  + y. fy = n Z

**Example :**

Z = 20 x2 + 23 y2 is a homogeneous function of degree 2

fx  = 40x ( first order partial derivative w.r.t. x. It has degree 1)

fy = 46y ( first order partial derivative w.r.t. y. Linear homogeneous function )

x. fx  + y. fy  = x. 40x + y.46y

 = 40x2 + 46 y2 = 2(20 x2 + 23 y2) = 2 Z

Thus we have verified Euler’s theorem for the given homogeneous function. Please remember that this theorem applies only to **homogeneous functions**.

**Marginal Productivities from a Production Function** :

We can calculate marginal productivities of labour and capital from a production function. where, quantity produced is a function of quantities of inputs used. Let us take a hypothetical case where there are on 2 factors of production , labour (L) and capital ( K )

The term “**marginal productivity**” refers to the extra output gained by adding one unit of labor; all other inputs are held constant. Marginal productivity of capital or MPK therefore gives marginal change in Output if we change only capital keeping labour constant

Similarly MPL  shows marginal change in output if we change only labour keeping capital constant

For example:

The production function for a commodity is

P =10*L* + 0.1*L*2 + 15*K* - 0.2*K*2 + 2*KL* where *L* is labour and *K* is Capital. P is the total product. (It can also be denoted by Q)



Rate of change of MPL and MPK can also be found out with partial derivatives:

MPL = 10 – 0.2L + 2K

fLL = - 0.2 (rate of change of MPL  w.r.t. L)

fLK = 2 (rate of change of MPL  w.r.t. K)

fKL = 2 (rate of change of MPK  w.r.t. L)

Notice that fLK = fKL

Generalized result: If Z = f(x y) then fxy = fyx  Second order cross partial derivatives are always equal to each other for all functions.

Euler’s Theorem for a production function :

Let us consider a homogeneous production function and verify Euler’s theorem

Q = 4LK + L2 + K2 . This is homogeneous function of degree 2.

L.MPL + K. MPk = L (4K + 2L) + K (4L+2L)

= 4LK + L2 + K2 + 4LK + L2 + K2

= 2(4LK + L2 +K2 ) = 2Q

Hence it is verified that L.MPL + K. MPk = 2 Q

**Cobb Douglas Production Function** is a special kind of production function which is of the form Q = A Lα Kβ .This function is based on the empirical study of the American manufacturing industry made by Paul H. Douglas and C.W. Cobb. It is a linear homogeneous production function of degree α+ β  and assumes that total output depends on only two factors of production, L and K. This function tells that output depends directly on L and K, and that part of output which cannot be explained by L and K is explained by A which is the ‘residual’, often called technical change. A, a and β are positive parameters where = > O, β > O.

Let us test the homogeneity and degree thereof.

Q = ALa Kβ

Suppose we multiply labour and capital quantities with a constant 2 ( or any other constant) then we can write

Q1 = f(2L,2K) = A (2L) α (2K) β

Q1 = (2) α + β (ALa Kβ ) = (2) α + β Q

Thus we can say that degree of homogeneity is α + β

There can be 3 scenarios:

**If α + β = 1** , then degree of homogeneity is 1. In such a case Q1 = (2) α + β Q = 2Q This means that doubling of L and K resulted in doubling of output. Such a function is therefore said to represent constant returns to scale.

 **If α + β < 1** . In this case doubling of labour and capital will result in production which is less than double. Such a case is called Decreasing Returns to scale.

**If α + β > 1.** In this casedoubling of labour and capital will result in production which is more than double. Such a case is said to represent Increasing Returns to scale.

Returns to scale is a long term phenomenon where all factors of production are increased by a constant proportion. For example, if a manufacturer increases his production from one factory to two factories in the long run, he is doubling his scale.

The returns to scale can be verified for any homogeneous production function and not just Cobb Douglas Production Function. For example consider this production function :

Q = L2 - 0.2K2 + 2KL

It can be verified that this production function is homogeneous of degree 2 and exhibits increasing returns to scale.